

A CONFORMAL INEQUALITY RELATED TO THE CONDITIONAL GAUGE THEOREM

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ABSTRACT. We prove the inequality $h(x)^{-1}G(x, y)h(y) \leq cG(x, y) + c$, where G is the Green function of a plane domain D , h is positive and harmonic on D , and c is a constant whose value depends on the topological nature of the domain. In particular, for the class of proper simply connected domains c may be taken to be an absolute constant. As an application, we prove the Conditional Gauge Theorem for plane domains of finite area for which the constant c in the above inequality is finite.

1. INTRODUCTION AND STATEMENT OF RESULTS

The study of conditioned Brownian motion in subdomains D of the plane leads to consideration of the ratios $G(x, y)h(y)/h(x)$, where G is the Green function of D and h is a positive harmonic function on D . For example, M. Cranston and the author [10] showed that if D has finite area and τ is the exit time from D of Brownian motion conditioned by the harmonic function h , then τ has finite expectation. This turns out to be equivalent to the inequality

$$(1.1) \quad \int_D G(x, y)h(y) dy \leq c|D|h(x),$$

where c is independent of D , h , and x , and $|D|$ denotes the area of D . A number of authors have considered variants, extensions, and applications of this inequality [3, 4, 6, 9, 12, 13, and 15]. The main purpose of this paper is to show that, for a large class of domains, (1.1) may be strengthened to the pointwise inequality,

$$(1.2) \quad h(x)^{-1}G(x, y)h(y) \leq cG(x, y) + c, \quad x, y \in D.$$

The class of domains includes all proper simply connected domains as well as a number of very pathological infinitely connected domains.

Inequality (1.2) is interesting because it shows how the boundary behaviors of G and h are related, and, like (1.1), it places constraints on the behavior of

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conditioned Brownian motion. To illustrate the latter point we will show how (1.2) implies the conditional gauge theorem for such domains.

Throughout the paper we assume D is a connected open subset of the plane and has a Green function in the classical sense, i.e., the Green function G is required to approach 0 at every boundary point. This is true, for example, if each connected component of the complement of D relative to the Riemann sphere \mathbf{C}^* contains at least 2 (hence, a continuum of) points. (Myrberg [17]. Also see [16, p. 179].)

Theorem 1.1. *Let $D \neq \mathbf{C}$ be a domain such that for some constant $\eta > 0$ we have $\text{diam}(K) \geq \eta$ for each connected component K of D^c . Then there is a constant c such that for all distinct x, y , and z in D ,*

$$(1.3) \quad G(x, z)^{-1} G(x, y) G(y, z) \leq c[G(x, y) + G(y, z)] + c.$$

For $D \neq \mathbf{C}$ simply connected the constant c in (1.3) may be taken to be an absolute constant. In the general case, c depends on the domain.

A related, but less precise, inequality was proved in [11] for bounded Lipschitz domains in d dimensions.

Because inequality (1.3) is conformally invariant, one may apply the theorem to any domain D for which the hypotheses are satisfied by some conformal image of D , provided one is careful to preserve the hypothesis $\text{diam}(K) \geq (\text{constant})$.

For the applications below it is useful to note that if $|D|$ is finite then c can be chosen to depend only on η and $|D|$ (see Remark 3.2).

Taking limits in (1.3) along sequences of z tending to ∂D we obtain (1.2) for harmonic functions h corresponding to minimal Martin boundary points. After cross multiplication by $h(x)$, the Martin representation shows that (1.2) holds for all $h > 0$ harmonic on D :

Corollary 1.2. *Let D be as in Theorem 1.1. Then for each positive harmonic function h on D we have*

$$h(x)^{-1} G(x, y) h(y) \leq c G(x, y) + c, \quad x, y \in D.$$

Here c is the same constant as in (1.3).

This result may be interpreted as a new kind of Harnack inequality in which the condition “ x and y belong to a fixed compact subset of D ” is replaced by a condition on $G(x, y)$:

Corollary 1.3. *Let D be as in Theorem 1.1 and $\lambda > 0$. Then there is a constant A having the same dependence on D as the constant c in (1.3) and which in addition depends on λ , such that for each positive harmonic function h on D ,*

$$G(x, y) = \lambda \Rightarrow A^{-1} h(x) \leq h(y) \leq A h(x).$$

The condition on the size of the boundary components of D is essential in each of the results above. In §5 we show that for bounded triply connected

domains the best constants c in (1.3) tend to infinity as η tends to zero. By connecting worse and worse such examples with very thin "tubes" one may easily construct an infinitely connected domain for which (1.3) fails.

Likewise, none of the above results extends to higher dimensions. Indeed, even the weaker result (1.1) fails for bounded simply connected domains in \mathbf{R}^d , $d \geq 3$ [10].

Following [11], we shall apply Theorem 1.1 to prove a conditional gauge theorem. Let $B(t)$ be Brownian motion and denote by τ_E the first exit time of $B(t)$ from the open set E . (If $E = D$ we will omit the subscript.) Let V be a Borel function on D and define the Feynmann-Kac functional $e_V(t) = \exp\{\int_0^t V(B(s)) ds\}$. The quantity $F(x) = E_x e_V(\tau)$ was introduced by Chung and Rao [8] who showed that for V bounded and $|D| < \infty$ one has the implication

$$(1.4) \quad F(x_0) < \infty \text{ for some } x_0 \in D \Rightarrow \sup_{x \in D} F(x) < \infty.$$

The E_x above denotes expectation relative to Brownian motion started from x . We shall write E_x^h for expectation relative to Doob's h -path Brownian motion started from x . When h corresponds to a minimal Martin boundary point z we will write this as E_x^z . In this case, the h -path process, also denoted $B(t)$, may be viewed as Brownian motion started from x and conditioned to converge to the point z as t tends to τ .

Let $F(x, z) = E_x^z e_V(\tau)$ whenever this is well defined. In [14] N. Falkner proved that for V bounded and D having C^2 boundary one has an implication analogous to (1.4):

$$(1.5) \quad F(x_0, z_0) < \infty \text{ for some } (x_0, z_0) \in D \times \partial D \\ \Rightarrow F(\cdot, \cdot) \text{ is bounded over } D \times \partial D.$$

The function F has been termed the *conditional gauge* and results of the form (1.5) *conditional gauge theorems* (CGT's). This result has been extended in a number of ways, and it has been applied to the study of the potential theory of the Schrödinger operator $-\frac{1}{2}\Delta + V$. (See [1 and 11].) Considerable effort has been expended towards obtaining the CGT for the widest possible class of domains with V belonging to the *Kato class* K_d , because of the fundamental role the latter plays in the theory of Schrödinger operators. We shall not define K_d here because we are interested in a wider (in the case of bounded domains) class given by (1.6) and (1.7) below. Instead we refer the reader to [1].

For $V \in K_d$ the best result to date is that (1.5) holds in the case of bounded Lipschitz domains [11]. In case $d = 2$ much more is true.

Theorem 1.4. *Let D be a domain having $|D| < \infty$ and otherwise satisfying the same hypotheses as in Theorem 1.1. Suppose V is a Borel function on D satisfying the following pair of conditions:*

$$(1.6) \quad (\text{Kato condition}) \quad \limsup_{r \rightarrow 0} \sup_{x \in D} \int_{D \cap B(x; r)} G(x, y) |V(y)| dy = 0,$$

and

$$(1.7) \quad V \in L^1(D)$$

Then the implication (1.5) is true.

We remark that each V in $L^p(D)$ for $p > 1$ satisfies (1.6) and (1.7), and that for bounded domains the class of functions satisfying these conditions includes the usual Kato class K_2 . The CGT fails in general for domains of infinite area. To construct a counterexample, take for D an infinite strip, for z one of the two Martin boundary points at infinity, and for V a sufficiently small constant.

It is also noteworthy that, while [11 and 15] convey the impression that the CGT in higher dimensions is related to smoothness properties of ∂D , our hypotheses are of a completely different character. One may even have $|\partial D| > 0$ —take for D the unit square, less all vertical segments of the form $\{x\} \times [1/4, 3/4]$, where x ranges over a Cantor subset of $[0, 1]$ of positive measure.

Finally, we should point out that the result of [11] is more general than (1.5) in that x_0 is allowed to lie in ∂D , with the conclusion being that F is bounded over $\overline{D} \times \overline{D}$. Our result does not extend in this way. Salisbury [18] has constructed a domain D satisfying our hypotheses, and points $x, z \in \partial D$, for which there does not exist a conditioned Brownian motion from x to z .

In the next section we prove Theorem 1.1 in the special case of a simply connected domain. Strictly speaking, this is unnecessary since the proof of the general case in §3 does not depend on having first established the result for simply connected domains. However, we feel that the proof is instructive, and moreover it leads to a reasonable value for the constant c (6.1 will work). Theorem 1.4 is proved in §4, and §5 contains the example alluded to in the discussion following Corollary 1.3.

2. THE SIMPLY CONNECTED CASE

Let D be a simply connected subset of \mathbb{C}^* . By performing an inversion if necessary we may assume that D is a proper simply connected subdomain of \mathbb{C} . We shall prove

$$(2.1) \quad G(x, z)^{-1} G(x, y) G(y, z) \leq c_1 [G(x, y) + G(y, z)] + c_2$$

with $c_1 = \frac{5}{2} \log 3$ and $c_2 = 8 \log 2 \log 3$.

By the Riemann mapping theorem we may assume D is the unit disc, $x = 0$, and $0 < y < 1$. In this case $G(y, z) = -\log |z - y| + \log |1 - z^* y|$, where z^* denotes the complex conjugate of z , and (2.1) reduces to

$$(2.2) \quad (\log |z|)^{-1} G(y, z) \log y \leq c_1 [-\log y + G(y, z)] + c_2.$$

Clearly, we may always assume $y < |z|$.

We shall consider the following four cases separately:

Case 1. $y < |z| < 1/2$,

Case 2. $|z| \geq 1/2$, $1/4 < y$, $|z|^2 < y$,

Case 3. $|z| \geq 1/2$, $1/4 < y \leq |z|^2$, and

Case 4. $|z| \geq 1/2$, $y \leq 1/4$.

In Case 1, first suppose $|y - z| \leq |z|/2$. Then $y \geq |z|/2$ so that $(\log|z|)^{-1}G(y, z)\log y \leq 2G(z, y)$. On the other hand, if $|y - z| > |z|/2$ then $-\log|y - z| \leq \log 2 - \log|z|$. Since $G(y, z) < \log(5/4) - \log|y - z|$ we conclude that $(\log|z|)^{-1}G(y, z)\log y < (\log 5)/\log 2 \log(1/y)$.

In case 2 we have

$$(\log|z|)^{-1}G(y, z)\log y \leq (\log|z|)^{-1}G(y, z)\log(|z|^2) = 2G(y, z).$$

As for Case 3, we will prove that

$$(2.3) \quad (\log|z|)^{-1}G(y, z)\log y \leq 8\log 2 \log 3.$$

It is well known that $G(z, y) \leq G(|z|, y)$. (This may be checked directly. It also follows from a general result about circularly symmetric domains [2, p. 154].) Therefore, in proving (2.3) we may assume that $1/4 < y \leq z^2 < z < 1$. From the Taylor expansion of $\log(1 - w)$ in powers of w we conclude that $-\log z \geq 1 - z$ and

$$\begin{aligned} -\log y &= (1 - y) + (1 - y)^2/2 + \cdots \\ &< (1 - y)[1 + (3/4)/2 + (3/4)^2/3 + \cdots] \\ &= (8/3)\log 2(1 - y). \end{aligned}$$

Expanding $G(y, z) = -\log(z - y) + \log(1 - zy)$ in powers of $(1 - z)$ gives

$$(2.4) \quad G(y, z) = (1 + y) \left(\frac{1 - z}{1 - y} \right) + \sum_{k=2}^{\infty} \left(\frac{1 - (-1)^k y^k}{k} \right) \left(\frac{1 - z}{1 - y} \right)^k.$$

Thus, with $c = (16/3)\log 2$ we have

$$\begin{aligned} (\log|z|)^{-1}G(y, z)\log y &< c \left(\frac{1 - y}{1 - z} \right) \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1 - z}{1 - y} \right)^n \\ &= c \sum_{n=0}^{\infty} \frac{1}{(n+1)} \left(\frac{1 - z}{1 - y} \right)^n \leq c \sum_{n=0}^{\infty} \frac{1}{(n+1)} \left(\frac{1 - z}{1 - z^2} \right)^n \\ &= c \sum_{n=0}^{\infty} \frac{1}{(n+1)} \left(\frac{1}{1 + z} \right)^n < c \sum_{n=0}^{\infty} (n+1)^{-1} \left(\frac{2}{3} \right)^n = \frac{3}{2} \log 3. \end{aligned}$$

Finally, turning to Case 4 we have $(-\log|z|)^{-1}G(y, z) \leq G(y, |z|)/(1 - |z|)$ and we may assume $z = |z|$. By (2.4)

$$G(y, z) \leq \left(\frac{5}{4} \right) \sum_{n=1}^{\infty} n^{-1} \left(\frac{1 - z}{1 - y} \right)^n.$$

Thus

$$G(y, z)/(1 - z) \leq \frac{5}{3} + \left(\frac{5}{3} \right) \sum_{n=1}^{\infty} (n+1)^{-1} \left(\frac{2}{3} \right)^n = \frac{5}{2} \log 3.$$

Multiplying both sides by $-\log y$ completes the proof.

3. PROOF OF THEOREM 1.1.

We begin with the following key lemma which enables us to assume without loss of generality that both $G(x, y)$ and $G(y, z)$ are large.

Lemma 3.1. *Let $\beta > 0$ and suppose the inequality*

$$(3.1) \quad G(x, z)^{-1} G(x, y) G(y, z) \leq c[G(x, y) + G(y, z)] + c$$

has been proved for a given domain D with $c = c(D) \geq 1$ and all distinct points x, y , and z in D which also satisfy

$$(3.2) \quad \min\{G(x, y), G(y, z)\} \geq \beta.$$

Then (3.1) holds for all distinct x, y , and z in D with c replaced by $c + 2\beta$.

Proof. Fix x, y , and z in D and suppose that (3.2) is false.

Case 1. $\max\{G(x, y), G(y, z)\} \geq \beta$.

By symmetry we may assume without loss of generality that $G(x, y) < \beta$ and $G(y, z) \geq \beta$. We will then show that

$$(3.3) \quad G(x, z)^{-1} G(x, y) G(y, z) \leq (c + \beta)G(y, z) + (c + \beta).$$

Let $A = \{v: G(v, y) \geq \beta\}$ and let ω denote the Brownian hitting distribution on A under P_x , i.e.,

$$\omega(dv) = P_x(B(\tau_A c) \in dv, \tau_A c < \tau).$$

Since $G(\cdot, y)$ is bounded and harmonic on $D \cap A^c$, we have

$$G(x, y) = E_x G(B(\tau_A c \wedge \tau), y) = \int_{\partial A} G(v, y) \omega(dv) = \beta \omega(A).$$

Now for any $v \in \partial A$ we have by (3.1) (which holds, by assumption, for the triple v, y, z),

$$(3.4) \quad \beta G(y, z) \leq c G(y, z) G(v, z) + (\beta + c) G(v, z).$$

Integration with respect to $\omega(dv)$ over ∂A gives

$$\begin{aligned} G(x, y) G(y, z) &= \beta \omega(A) G(y, z) \\ &\leq c G(z, y) \int_{\partial A} G(v, z) \omega(dv) + (\beta + c) \int_{\partial A} G(v, z) \omega(dv). \end{aligned}$$

But

$$\int_{\partial A} G(v, z) \omega(dv) = E_x G(B(\tau_A c \wedge \tau), z) \leq G(x, z).$$

Substituting this in the inequality above and then dividing through by $G(x, z)$, we obtain (3.3).

Case 2. $\max\{G(x, y), G(y, z)\} < \beta$.

We may repeat the argument just given, only using $c + \beta$ in place of c in (3.4). The proof of Lemma 3.1 is complete.

Next, we show that there exists a constant $\beta = \beta(D, \eta)$ such that

$$(3.5) \quad |u - v| > \eta/8 \Rightarrow G(u, v) < \beta.$$

To see this, we may assume that D^c contains an unbounded component, by performing a suitable inversion if necessary. Therefore the probabilistic formula $G(u, v) = E_u \log(|v - B(\tau)||u - v|^{-1})$ is valid. We may assume without loss of generality that $u = 0$. Define the Brownian maximal function by $B^*(\tau) = \sup\{|B(t)| : 0 \leq t < \tau\}$. The results of [5] imply that $B^*(\tau)$ has a finite p th moment for each $0 < p < 1/2$. To estimate this moment, note that the circular symmetrization, D^* , of D with respect to the origin is contained in the plane minus a half-line on the negative x -axis of the form $(-\infty, x_0]$. Let σ be the hitting time of Brownian motion to this half-line. Then using the results of [2] on the behavior of harmonic measure under symmetrization, as in [5, p. 199], we obtain

$$E_0[B^*(\tau)]^p \leq E_0[B^*(\sigma)]^p \leq C,$$

where $C = C(p, x_0)$, provided $0 < p < 1/2$. Thus, for any such p ,

$$\begin{aligned} G(0, v) &< (1/p)|v|^{-p} E_0|v - B(\tau)|^p \leq (1/p)|v|^{-p} \{|v|^p + E_0[B^*(\tau)]^p\} \\ &\leq (1/p)(C + |v|^p)|v|^{-p}, \end{aligned}$$

and statement (3.5) follows.

Remark 3.2. If $|D| < \infty$, one may estimate more explicitly as follows:

$$\begin{aligned} G(0, v) &< (1/2)|v|^{-2} E_v|B(\tau)|^2 = 1/2 + |v|^{-2} E_v \tau \\ &\leq 1/2 + c|D||v|^{-2} \end{aligned}$$

where c is an absolute constant. (See, e.g., [10, Lemma 2.1] for the last inequality.) Thus in this case we may choose β depending only on $|D|$ and η . (We shall ultimately need to increase the value of β further for other purposes below.)

To prove (1.3) we may assume $y = 0$ and $|z| \geq |x| = x$. By Lemma 3.1 we may further assume $\min\{G(x, 0), G(0, z)\} \geq \beta$ and, therefore

$$(3.6) \quad 0 < x < \eta/8.$$

Let $B = B(0; 2x)$ and $B^\sim = B(0, 4x)$. It follows from the maximum principle and the connectivity of D that for any $\lambda > 0$ the set $\{w : G(w, z) > \lambda\}$ is connected. Taking $\lambda = G(0, z)/2$ shows that it is possible to find a curve Γ connecting 0 and z such that

$$(3.7) \quad G(w, z) \geq G(0, z)/2, \quad w \in \Gamma.$$

(Refer to Figure 1.)

It is now time to specify the final value of β . Take β to satisfy $\beta > \beta_0$, where β_0 is the value at the point $(1, 0)$ of the Green function of the slit plane $\mathbb{C} \setminus [-4, -2]$, in addition to the earlier constraints placed upon β .

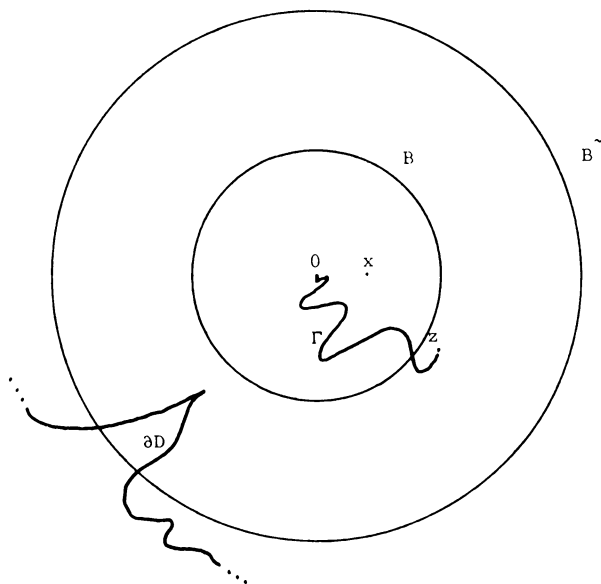


FIGURE 1

Claim. If K is a connected component of D^c then $G(0, x) > \beta$ implies that K does not meet B , i.e., $B \subset D$.

Leaving aside, for a moment, the verification of the claim, let us show how (1.3) follows. Indeed, we will prove $G(x, z)^{-1}G(x, 0)G(0, z) \leq cG(x, 0)$, a stronger inequality. Let μ denote the hitting time to Γ , and ν the exit time from B , of Brownian motion. Then $P_x(\mu < \tau) \geq P_x(\mu < \nu)$. We may use the theory of symmetrization to estimate the latter probability from below. Let μ^* denote the hitting time of Brownian motion to the segment $[-x, 0]$. Then by [2, Theorem 7] and the fact that $|z| \geq x$, we have

$$P_x(\mu < \nu) \geq \inf_{|\xi|=x} P_\xi(\mu^* < \nu) \stackrel{d}{=} p_0 > 0,$$

where p_0 is an absolute constant. By superharmonicity of G ,

$$G(x, z) \geq E_x G(B(\mu \wedge \tau), z) = E_x [G(B(\mu), z); \mu < \tau] \geq p_0 G(0, z)/2.$$

Thus we conclude that $G(x, z)^{-1}G(x, 0)G(0, z) \leq (2/p_0)G(x, 0)$, which is the desired result.

There remains only to prove the claim, which we do by contradiction. Suppose $K \cap B$ were not empty, say $w_1 \in K \cap B$. By (3.6) and the definition of η we have that K cannot be contained in B^\sim (else its diameter would be too small) so we can find a point w_2 in K which lies outside the disc B^\sim . Since K is connected, there is some curve γ contained in K which connects w_1 to w_2 . It follows that the circular symmetrization D^* is contained in the domain R formed by removing the segment $[-4x, -2x]$ from the plane. By dilation invariance of G_R we then have $G_R(x, 0) = \beta_0$. Then, combining Theorem 5

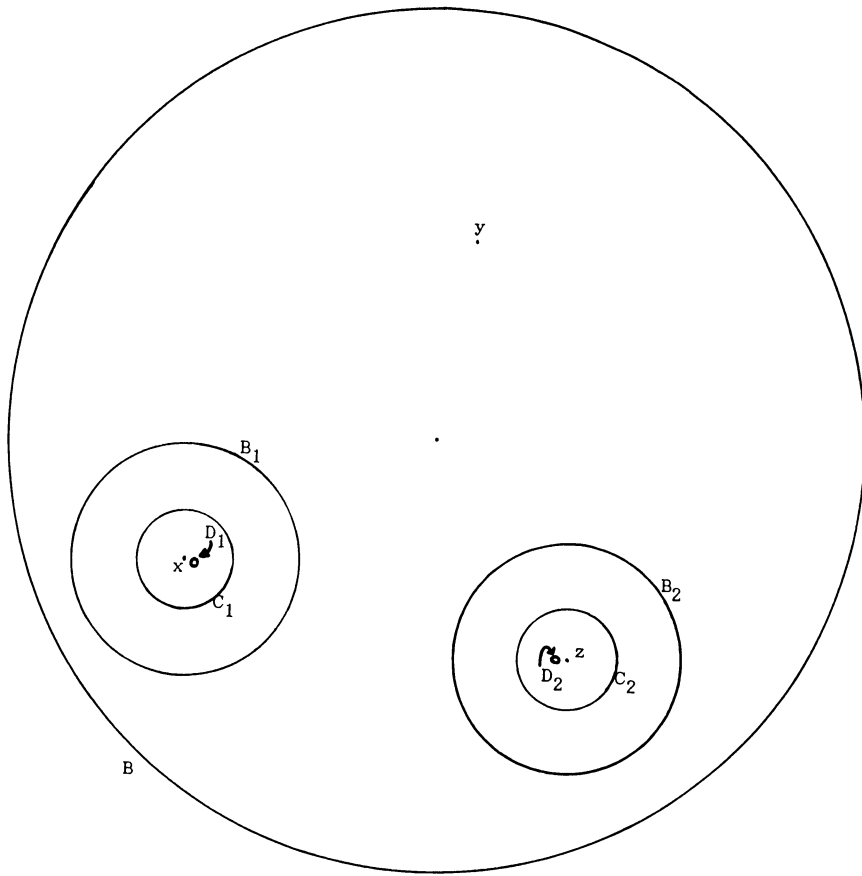


FIGURE 2

and the Corollary of Proposition 5 in [2], we have

$$G(x, 0) \leq G_{D^*}(x, 0) \leq G_R(x, 0) = \beta_0 < \beta,$$

a contradiction which completes the proof of Theorem 1.1.

4. PROOF OF THEOREM 1.4

Let h be a positive harmonic function on D . In [7] Chung shows how to reduce the CGT to the “gauge theorem” of [19] whenever the following two conditions hold true:

(A) $P_x^h(\tau < \infty) = 1$, and,
for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

(B) $\int_E h(x)^{-1} G(x, y) h(y) |V(y)| dy < \varepsilon$
for each open subset E of D satisfying $|E| < \delta$.

Since the gauge theorem of [19] is true in our setting, we need only verify (A) and (B) to prove Theorem 1.4. (Alternatively, one may follow the proof of [15, Theorem 2].) However, Chung’s argument that (A) and (B) imply CGT is given

under the (inessential) assumption that D be bounded Lipschitz. Therefore, before proving (A) and (B), we will indicate the necessary changes in Chung's argument.

The crux of the matter is the following: For a positive number s let $D_0 = \{x \in D: \text{dist}(x, D^c) > s/2\}$. Note that D_0 has compact closure since we are assuming that D has finite area. The open set $\{x \in D: \text{dist}(x, D^c) < s\}$ has finitely many connected components since each must contain some disc of radius $s/2$. By joining these components with very thin tubes contained in D , it is easy to construct a set E with the following properties:

- (4.1) E is open and connected,
- (4.2) $|E|$ may be made small by taking s to be small,
- (4.3) E contains ∂D_0 , and
- (4.4) D_0 contains $(\partial E) \cap D$.

Our proof of the next result essentially follows Chung, except for the use of Theorem 1.1.

Lemma 4.1. *There exists a constant b depending only on s and D such that for any $h > 0$ harmonic on D we have*

$$(4.5) \quad P_x^h(\tau_E = \tau) \geq b > 0, \quad x \in \partial D_0.$$

Proof. Fix a reference point z_0 in D_0 . Let $\partial_m D$ denote the minimal Martin boundary of D and $K(x, y)$ the Martin kernel normalized by $K(z_0, \cdot) \equiv 1$ on $\partial_m D$. Recall that $\partial_m D$ is compact in the Martin topology and K is continuous on $\partial D_0 \times \partial_m D$ in the (euclidean) \times (Martin) topology. (See, e.g., [16, Lemma 12.3].)

It is sufficient to prove (4.5) for $h(\cdot) = K(\cdot, z)$, $z \in \partial_m D$. Now

$$P_x^z(\tau_E = \tau) = K(x, z)^{-1} [K(x, z) - f(x, z)]$$

where

$$f(x, z) = E_x[K(B(\tau_E), z); \tau_E < \tau].$$

For any point y in the closure of D_0 we have for some sequence z_n

$$\begin{aligned} K(y, z) &= \lim_{z_n \rightarrow z} G(y, z_n)/G(z_0, z_n) \\ &\leq \limsup_{n \rightarrow \infty} c[G(y, z_n)/G(z_0, y) + 1] + cG(z_0, y)^{-1} \\ &= c + cG(z_0, y)^{-1}, \end{aligned}$$

where the inequality follows from Theorem 1.1. The latter quantity is bounded as y ranges over the closure of D_0 . Since $B(\tau_E) \in (\partial E) \cap D$ on $\{\tau_E < \tau\}$, it follows from (4.4) and the dominated convergence theorem that $f(x, \cdot)$ is continuous on $\partial_m D$ for each x in D ; moreover, $f(\cdot, z)$ is harmonic, hence continuous on E , for each z in $\partial_m D$. Since $\partial_m D$ and the closure of D_0 are disjoint compact sets, it follows that f is continuous on $\partial D_0 \times \partial_m D$.

Finally, $K(x, z) - f(x, z)$ is strictly positive on E for each z in $\partial_m D$. (This follows from its harmonicity on E , since f is bounded on E while K is unbounded there—see [10, Lemma 2.2].) The desired result now follows by continuity.

The remainder of Chung's argument, involving a sequence of hitting times, goes through without any essential change. (His notation is somewhat different from ours.)

There remains to verify (A) and (B). Condition (A) is immediate from the main result of [10] for any *arbitrary* domain of finite area. As for (B), first choose $r > 0$ so small that

$$\int_{B(x; r)} G(x, y) |V(y)| dy < \frac{\varepsilon}{2c}, \quad x \in D,$$

where c is as in Theorem 1.1, and we assume G is extended to be equal to zero off D . For x, y in D , $|x - y| > r$, there is a constant M such that $G(x, y) \leq M$. (See (3.5) above.) By (1.7) it is possible to write $V = V_1 + V_2$ where V_2 is bounded and $\|V_1\|_1$ is as small as desired. Then, by Corollary 1.2,

$$\begin{aligned} \int_E h(x)^{-1} G(x, y) h(y) |V(y)| dy \\ &\leq c \int_E G(x, y) |V(y)| dy + c \int_E |V(y)| dy \\ &\leq \varepsilon/2 + (M + 1)c \int_E |V(y)| dy \\ &\leq \varepsilon/2 + c(M + 1) |E| \|V_2\|_\infty + c(M + 1) \|V_1\|_1. \end{aligned}$$

The latter expression may be made smaller than ε by first taking $\|V_1\|_1$, and then $|E|$, to be sufficiently small.

5. AN EXAMPLE

Let x, y , and z be the vertices of an equilateral triangle centered at 0 and $B = B(0; R)$ a disc centered at 0 and large enough to contain x, y and z ; say, $R > 2|x|$ for definiteness. We will construct a domain D of the form $D = B \setminus (D_1 \cup D_2)$, where D_1 and D_2 are small discs such that the constants in (1.3) can be made as large as desired by taking R large and the D_i very small and very close to x and z . The discs B_i and C_i shown in Figure 2 are not part of the description of D , but will be used in the argument. The intuition behind the example is that the value of each of $G(x, y)$ and $G(y, z)$ is significantly affected (lowered) by only one of the D_i , whereas $G(x, z)$ is affected by both.

Let $B_1 = B(x; |x|/2)$ and $B_2 = B(z; |z|/2)$ and choose discs C_i concentric with the B_i such that

$$(5.1) \quad \text{diam}(C_i) < \text{diam}(B_i)/2, \quad i = 1, 2,$$

and so small that

$$(5.2) \quad G_B(u, v) \leq 2G_{B \setminus (C_1 \cup C_2)}(u, v), \quad u, v \in B \setminus (B_1 \cup B_2).$$

The following inequalities, in which c stands for a constant depending on R and the choice of the C_i , are all immediate consequences of Harnack's inequality:

$$\begin{aligned}
 & G_{B \setminus (C_1 \cup C_2)}(u_1, v) \leq c G_{B \setminus (C_1 \cup C_2)}(u_2, v), \\
 & \quad v \in \text{cl}(B_2) \text{ (resp. } B_1), u_1, u_2 \in \partial B_1 \text{ (resp. } \partial B_2), \\
 (5.3) \quad & G_{B \setminus (C_1 \cup C_2)}(u_1, y) \leq c G_{B \setminus (C_1 \cup C_2)}(u_2, y), \quad u_i \in \partial B_1 \text{ (resp. } \partial B_2), \\
 & \frac{1}{c} G_B(x, y) \leq G_B(u, y) \leq c G_B(x, y), \quad u \in \partial B_1, \quad \text{and} \\
 & \frac{1}{c} G_B(z, y) \leq G_B(v, y) \leq c G_B(z, y), \quad v \in \partial B_2.
 \end{aligned}$$

Notice that c need not increase as R increases or as the diameters of the C_i decrease.

Let D_i ($i = 1, 2$) be small discs contained in C_i with $x \notin D_1$ and $z \notin D_2$. Let $p_1 = P_x(B(t))$ does not hit D_1 before leaving B_1 , and $p_2 = P_z(B(t))$ does not hit D_2 before leaving B_2 . Then the p_i can be made as small as desired by taking D_1 and D_2 sufficiently close to x and z , respectively.

Let $D = B \setminus (D_1 \cup D_2)$ and $G = G_D$.

By (5.2) and monotonicity of the Green function in its domain,

$$(5.4) \quad G_B(u, v) \leq 2G(u, v), \quad u, v \in B \setminus (B_1 \cup B_2).$$

Next, we prove the following inequalities, in which c is the constant of (5.3):

$$(5.5) \quad (p_1/2c^2)G_B(x, y) \leq G(x, y) \leq c^2 p_1 G_B(x, y),$$

$$(5.6) \quad (p_2/2c^2)G_B(y, z) \leq G(y, z) \leq c^2 p_2 G_B(y, z), \quad \text{and}$$

$$(5.7) \quad (p_1 p_2/2c^4)G_B(x, z) \leq G(x, z) \leq p_1 p_2 c^4 G_B(x, z).$$

Let μ and ν be the Brownian exit times from B_1 and B_2 , respectively. Also choose and fix arbitrary points $x_i \in \partial B_i$ ($i = 1, 2$). Then

$$\begin{aligned}
 G(x, y) &= E_x G(B(\mu \wedge \tau), y) \geq (p_1/c)G(x_1, y) \\
 &\geq (p_1/2c)G_B(x_1, y) \geq (p_1/2c^2)G_B(x, y),
 \end{aligned}$$

which is the left-hand side of (5.5). For the other side we have

$$\begin{aligned}
 G(x, y) &= E_x G(B(\mu \wedge \tau), y) \leq c p_1 G(x_1, y) \\
 &\leq c p_1 G_B(x_1, y) \leq c^2 p_1 G_B(x, y).
 \end{aligned}$$

The proof of (5.6) is the same.

To obtain the right-hand side of (5.7) we employ the same reasoning twice in succession:

$$\begin{aligned}
 G(x, z) &= E_x G(B(\mu \wedge \tau), z) \leq c p_1 G(x_1, z) \leq c p_1 E_z G(x_1, B(\nu \wedge \tau)) \\
 &\leq c^2 p_1 p_2 G(x_1, x_2) \leq c^2 p_1 p_2 G_B(x_1, x_2) \\
 &\leq c^4 p_1 p_2 G_B(x, z).
 \end{aligned}$$

The proof of the other side of (5.7) is similar.

Now, combining (5.5)–(5.7), we have

$$\begin{aligned} G(x, z)^{-1} G(x, y) G(y, z) &\geq (1/4c^8) G_B(x, z)^{-1} G_B(x, y) G_B(y, z) \\ &= (1/4c^8) G_B(x, y), \end{aligned}$$

since $G_B(x, y) = G_B(y, z) = G_B(x, z)$ by symmetry.

The desired conclusion follows since $G_B(x, y)$ can be made arbitrarily large by taking R large, and the ratio $G_B(x, y)/G(x, y)$ can be made arbitrarily large by ensuring that p_1 is small.

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